Results of Wave Interferences.

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The combined influences of action and reaction, elasticity, density, and fundamental velocity, in the arrangement of the solar system, are shown by the symmetrical formula,

$$\left(\frac{\mu + \mu_1}{\mu_1}\right)^{\frac{\lambda_1 + \lambda}{\lambda}} = \frac{\rho_1 + \rho}{\rho} \qquad (1.)$$

in which μ = mass of Sun ; μ_1 = mass of Jupiter ; λ = average velocity of complete solar dissociation = $214.86\ \rho \div 497.825$ = velocity of light ; $\lambda_1 = 2\sqrt{g\rho}$ = 2 × velocity of incipient solar dissociation = mean radial velocity of complete solar dissociation = $\frac{4 \times (214.86)^{\frac{3}{2}}\ \pi \rho}{\text{No. seconds in 1 year}}$; ρ_1 = Jupiter's projectile radius or mean perihelion distance from Sun ; ρ = Sun's equatorial radius. Substituting the values $\lambda = \rho \div 2.317$; $\lambda_1 = \rho \div 344.15$; $\rho_1 = 1069.62$; $\rho = 1$; the equation reduces to

$$\left(\frac{\mu + \mu_1}{\mu_1}\right)^{1.0029} = 1070.62 \quad \therefore \quad \frac{\mu + \mu_1}{\mu_1} = 1049.24.$$

Bessel's estimate is 1048.88 ; the difference between the theoretical and the observed value being only $\frac{1}{29}$ of 1 per cent.

The velocity of light also appears as an important factor in the following equations, thus furnishing further evidence, both of the significance of Earth's position, at the centre of the belt of greatest condensation, and of Jupiter's influence :

$$\frac{(n\pi)^3 \sqrt{fr}}{\Delta} = \lambda \qquad (2)$$

$$\frac{\mu}{\mu_2} = \left(\frac{\lambda}{\lambda_1}\right)^2 \times 2^{\frac{3}{2}} \qquad (3)$$

$$\frac{\delta}{\rho} = \frac{\lambda}{\lambda_1} \times 2^{\frac{2}{3}} \qquad (4)$$

$$t_2 = \frac{\lambda}{\lambda_1} \times 1.061 \text{ days.} \qquad (5)$$

$$t : t_1 : : \lambda_1 : \lambda \qquad (6)$$

In these equations $n\pi \sqrt{fr}$ = terrestrial dissociative velocity ; μ_2 =

mass of Earth ; $\triangle$ = density of Earth in units of Sun's density ; $2^{\frac{3}{2}}$ = time of revolution at $2r$; $2^{\frac{2}{3}}$ = radius of revolution for $2t$; δ = Earth's mean distance from Sun ; t = time of oscillation through major-axis equivalent to Sun's possible atmosphere, or to $\frac{1}{3}$ of Earth's radius vector ; t_1 = time of Jupiter's revolution ; t_2 = time of Earth's revolution ; 1.061 = Jupiter's $\frac{\text{secular aphelion}}{\text{mean}}$ radius vector.

It is evident, from equation (6), that $\frac{t + t_1}{t_1}$ might be substituted for $\frac{\lambda_1 + \lambda}{\lambda}$, in the exponent of equation (1).

In the undulations which are generated by the two controlling masses, μ and μ_1, we may naturally look for harmonic interferences, not only in the light spectrum, but also in cosmical aggregations and in elementary molecular groupings. If we compare μ and μ_1 at Jupiter's present perihelion, we find that the product of Jupiter's radius vector by its mass is 1.0153 times the product of Sun's radius by its mass. Representing 1.0153 by n, and taking $a = 6 \times .0153 = .0918$, the harmonic progression, $\frac{1}{n + a}$, $\frac{1}{n + 2a}$, $\frac{1}{n + 3a}$, etc., gives us the following nodal divisors and approximations, in millionths of a millimetre, to wave-lengths of Fraunhofer lines :

Denominators.	Nodal Divisors.	Quotients.	Observed.
1	1.0000	761.20	A 761.20
$n + a$ (*f*)	1.1071	687.56	B 687.49
	[1.1530	660.19]	C 656.67
$n + 2a$	1.1989	634.92	
$n + 3a$ (*f*)	1.2907	589.76	D 589 74
$n + 4a$	1.3825	550.60	
	[1.4437	527.26]	E 527.38
$n + 5a$	1.4743	516.31	*b* 517.70
$n + 6a$ (*f*)	1.5661	486.05	F 486.52
$n + 7a$	1.6579	459.13	
$n + 8a$	1.7497	435.05	G 431.03
	[1.7650	431.27]	
$n + 9a$	1.8415	413.37	
	[1.9180	396.87]	H 397.16
$n + 10a$ (*f*)	1.9333	393.73	H¹ 393.59

The harmonic interferences indicated by the series marked (*f*) are the most interesting, both on account of the closeness between the theoretical quotients and the corresponding observed values, and because the successive denominator increments, are figurate.

Of the remaining six lines, three (A, *b*, G,) approximate so closely to the corresponding harmonic quotients, the greatest deviation being less than one per cent., that they may be properly regarded as illustrations of secondary interferences; introducing two harmonic triplets, with a uniform denominator difference of $2\,a$, ($n + 2\,a$, $4\,a$, $6\,a$; $n + 6\,a$, $8\,a$, $10\,a$).

The bracketed divisors indicate tertiary harmonics, based on denominator differences of $a' = .0153$: $1.1530 = 1 + 10\,a'$; $1.4437 = 1 + 29\,a'$; $1.7650 = 1 + 50\,a'$; $1.9180 = 1 + 60\,a'$. The greatest difference between the theoretical and observed values is less than $\frac{5}{9}$ of one per cent.; the other differences range between $\frac{1}{44}$ and $\frac{1}{13}$ of one per cent.

Among the subordinate spectral lines there are some as I have shown elsewhere,* which are closely represented by the denominators $n + 2\,a$, $n + 4\,a$, $n + 5\,a$, $n + 7\,a$, $n + 8\,a$, $n + 9\,a$. But, on account of the great number of faint lines, such accordances are less satisfactory than those which can be found in the lines which are more widely separated and more prominent.

In planetary aggregation the interference waves have manifested their influence most strikingly at luminous internodes. The denominators are exponential, indicating roots which are to be extracted, instead of divisions which are to be made. It will be noticed that the first six exponential denominators in the following table, are arithmetical means between the adjacent numbers in the primitive series of nodal divisors in the foregoing table, and that the others are formed by successive denominator increments of $\frac{5}{4}\,a$.

Exponential Denominators.	Roots.	Observed.	
1.0000	6453	6453	Neptune.
1.0536	4130	4122	Uranus.
1.1530	2015	2050	Saturn.
1.2448	1150	1118	Jupiter.
1.3366	708	728	Freia.
1.4284	465	473	Flora.
1.5202	321	327	Mars.
1.6350	214	215	Earth.
1.7497	150	155	Venus.
1,8644	111	110	Ven.-Mer.
1.9792	84	83	Mercury.
2.0939	66	64	Mercury, s. p.
2.2089	53	53	Mercury, c. o.

The "observed" values are the mean planetary vector-radii, in units of Sun's radius. "Ven.-Mer." is the arithmetical mean between Venus's mean distance (155) and Mercury's secular perihelion (64). "Mercury, c. o." is the centre of spherical oscillation ($_{1}$ .4) of a nebula extending to Mercury's mean distance.

The harmonic interferences in the spectra of chemical elements may probably be best studied, by beginning with those which contain few

* *Ante*, p. 110.

prominent lines. The wave-measurements, in all of the following comparisons, are taken from the paper of Professor Wolcott Gibbs, in the American Journal of Science, second series, vol. xlvii, pp. 198, seq. Kirchhoff's lines are indicated by K; Huggins's by H; Gibbs's groupings of corresponding lines, in the observations of both Kirchhoff and Huggins, by K H; the left-hand columns containing Kirchhoff's estimates, and the right-hand columns those of Huggins:

MERCURY, K H.

Wave-lengths.		Quotients.		Theoretical.	
568.47	568.55	1.0000	1.0000	1.0000	1
546.33	546.13	1.0407	1.0411	1.0406	$1 + 6a$
542.80	542.80	1.0473	1.0484	1.0474	$1 + 7a$

LEAD, K H.

Wave-lengths.		Quotients.		Theoretical.	
561.29	561.46	1.0000	1.0000	1.0000	1
537.71	537.85	1.0439	1.0439	1.0440	$1 + 3a$
439.07	438.93	1.2784	1.2792	1.2784	$1 + 19a$

LITHIUM, H.

Wave-lengths.	Quotients.	Theoretical.	
610.73	1.0000	1.0000	1
479.48	1.2277	1.2214	$1 + 2a$
459.93	1.3279	1.3321	$1 + 3a$

RUTHENIUM AND IRIDIUM, K.

Wave-lengths.	Quotients.	Theoretical	
635.45	1.0000	1.0000	1
545.44	1.1650	1.1646	$1 + 5a$
530.52	1.1973	1.1975	$1 + 6a$

CHROMIUM, K.

Wave-lengths.	Quotients.	Theoretical.	
541.35	1.0000	1,0000	1
521.20	1.0387	1.0387	$1 + 111a$
520.98	1.0391	1.0391	$1 + 112a$
520.83	1.0394	1.0394	$1 + 113a$

COPPER, K.

Wave-lengths.	Quotients.	Theoretical.	
578.67	1.0000	1.0000	1
529.30	1.0933	1.0914	$1 + 6a$
522.24	1.1070	1.1066	$1 + 7a$
465.64	1.2428	1.2437	$1 + 16a$

Arsenic, K H.

Wave-lengths.		Quotients.		Theoretical.	
617.54	617.67	1.0000	1.0000	1.0000	1
611.69	611.67	1.0096	1.0098	1.0093	$1 + a$
578.95	578.73	1.0667	1.0673	1.0650	$1 + 7\ a$
533.55	533.41	1.1566	1,1580	1.1579	$1 + 17\ a$

Magnesium, K.

Wave-lengths.	Quotients.	Theoretical.	
518.73	1.0000	1.0000	1
517.64	1.0021	1.0020	$1 + 2\ a$
517.17	1.0030	1.0030	$1 + 3\ a$
459.62	1.1286	1.1285	$1 + 9\ b$
448.57	1.1564		
448.39	1.1569	1.1570	$1 + 11\ b$

Tin, K H.

Wave-lengths.		Quotients.		Theoretical.	
645.83	645.27	1.0000	1.0000	1.0000	1
615.59		1.0491		1.0530	$1 + a$
556.83		1.1598		1.1590	$1 + 3\ a$
556.59		1.1604		1.1620	$1 + 2\ b$
510.55	510.40	1.2650	1.2642	1.2650	$1 + 5\ a$
459.47		1.4056		1.4050	$1 + 5\ b$
453.41		1.4244		1.4240	$1 + 8\ a$

Potassium, H.

Wave-lengths.	Quotients.	Theoretical.	
630.85	1.0000	1.0000	1
624.81	1.0097	1.0097	$1 + \frac{1}{3}\ a$
613.25	1.0287	1.0291	$1 + a$
583.78	1.0806	1.0802	$1 + c$
581.79	1.0843	1.0843	$1 + b$
580.80	1.0862	1.0872	$1 + 3\ a$
551.96	1.1430	1.1454	$1 + 5\ a$
483.18	1.3360	1.3371	$1 + 4\ b$
438.96	1.4372	1.4362	$1 + 15\ a$
431.16	1.4632	1.4653	$1 + 16\ a$
426.00	1.4809	1.4810	$1 + 6\ c$
418.77	1.5064	1.5057	$1 + 6\ b$

Silver, K H.

Wave-lengths.		Quotients.		Theoretical.	
547.55	547.44	1.0000	1.0000	1.0000	1
546.96	546.63	1.0011	1.0015	1.0013	$1 + a$
521.32	521.34	1.0503	1.0501	1.0502	$1 + 38\ a$

This cannot be regarded as a satisfactory accordance.

Zinc, K H.

Wave-lengths.		Quotients.		Theoretical.	
636.99	637.37	1.0000	1.0000	1.0000	1
610.64	610.89	1.0432	1.0442	1.0390	$1 + a$
589.90	589.90	1.0798	1.0805	1.0781	$1 + 2\,a$
472.25	471.98	1.3488	1.3504	1.3513	$1 + 9\,a$

Cadmium, K H.

Wave-lengths.		Quotients.		Theoretical.	
647.22	647.08	1.0000	1.0000	1.0000	1
644.59		1.0041		1.0041	$1 + a \div 28$
531.27	531.01	1.2182	1.2186	1.2300	$1 + 2\,a$
509.00	508.83	1.2715	1.2717	1.2727	$1 + 5\,b$
480.56	480.27	1.3468	1.3473	1.3450	$1 + 3\,a$
468.10		1.3826		1.3818	$1 + 7\,b$
441.94	441.81	1.4645	1.4646	1.4600	$1 + 4\,a$

The quotient of Kirchhoff's sixth wave-length by the seventh (468.10 $\div$ 441.94), is equal to the quotient of the fourth by the fifth (509 $\div$ 480.56 $=$ 1.0592). The harmonic denominators, $1 + 7\,c$, $1 + 11\,c$, $1 + 15\,c$—if $c = 311.6$—give 1.2181, 1.3428, 1.4674 ; but this is not so satisfactory a representation, on the whole, as the one I have adopted. $(2 + 3 + 4)\,a = (5 + 2 \times 7)\,b$.

Lanthanum, K.

Wave-lengths.	Quotients.	Theoretical.	
538.56	1.0000	1.0000	1
538.43	1.0003	1.0003	$1 + \frac{1}{4}\,a$
538.00	1.0011	1.0011	$1 + a$
534.48	1.0077	1.0077	$1 + 7\,a$
520.80	1.0341	1.0340	$1 + 31\,a$
519.20	1.0373	1.0373	$1 + 34\,a$
518.69	1.0383	1.0384	$1 + 35\,a$
481.59	1.1183	1.1183	$1 + 108\,a$

Sodium, H.

Wave-lengths.	Quotients.	Theoretical.	
616.74	1.0000	1.0000	1
616.56	1.0002		
590.04	1.0452	1.0455	$1 + 6\,a$ $(6 = 1 + 5)$
589.43	1.0462		
569.46	1.0830	1.0835	$1 + 11\,a$ $(11 = 1 + 2 \times 5)$
568.90	1.0840		
515.90	1.1954		
515.37	1.1966	1.1973	$1 + 26\,a$ $(26 = 1 + 5 \times 5)$
498.87	1.2362	1.2362	$1 + 31\,a$ $(31 = 1 + 6 \times 5)$

ANTIMONY, K H.

Wave-lengths.		Quotients.		Theoretical.	
630.84	630.49	1.0000	1.0000	1.0000	1
613.50	613.73	1.0283	1.0273	1.0270	$1 + 2a$
598.41	598.72	1.0542	1.0531	1.0540	$1 + 4a$
591.61	591.45	1.0663	1.0660		
589.76	589.76	1.0697	1.0691	1.0675	$1 + 5a$
564.54	564.41	1.1174	1.1171	1.1165	$1 + 3b$
557.19	557.18	1.1322	1.1316	1.1350	$1 + 10a$
546.61	546.33	1.1554	1.1540	1.1553	$1 + 4b$
471.10	471.03	1.3391	1.3385	1.3375	$1 + 25a$

ARSENIC, K.

Wave-lengths.	Quotients.	Theoretical.	
617.54	1.0000	1.0000	1
611.69	1.0096	1.0093	$1 + a$
603.38	1.0235	1.0244	$1 + 2b$
578.95	1.0666	1.0653	$1 + 7a$
558.29	1.1063	1.1096	$1 + 9b$
550.42	1.1219	1.1217	$1 + 10b$
538.75	1.1462	1.1461	$1 + 12b$
533.55	1.1574	1.1585	$1 + 17a$
521.32	1.1846	1.1826	$1 + 15b$

The sixth quotient is also very nearly $1.1212 = 1 + 13a$; or $13a = 10b$.

BARIUM, K H.

Wave-lengths.		Quotients.		Theoretical.	
650.24	650.44	1.0000	1.0000	1.0000	1
611.75	612.15	1.0629	1.0625	1.0634	$1 + 4a$
603.08	602.70	1.0782	1.0792	1.0792	$1 + 5a$
597.05	597.58	1.0891	1.0885	1.0890	$1 + 15c$
585.51	585.67	1.1106	1.1106	1.1109	$1 + 7a$
582.88	582.77	1.1156	1.1161	1.1159	$1 + 2b$
578.51	578.00	1.1240	1.1253	1.1246	$1 + 21c$
553.95	554.06	1.1738	1.1740	1.1739	$1 + 3b$
552.40	552.06	1.1771	1.1782	1.1780	$1 + 30c$
493.78	493.57	1.3168	1.3178	1.3168	$1 + 20a$
490.20	490.23	1.3265	1.3268	1.3264	$1 + 55c$

The eighth quotient is also very nearly $1 + 11a = 1.1742$; or $113 = 3b$.

STRONTIUM, K H.

Wave-lengths.		Quotients.		Theoretical.	
641.38	641.39	1.0000	1.0000	1.0000	
553.90	553.74	1.1579	1.1583	1.1592	$1 + d$
552.57	552.38	1.1607	1.1614	1.1610	$1 + e$
550.83	550.61	1.1645	1.1649	1.1647	$1 + aa$

STRONTIUM, K. H.—*Continued.*

Wave-lengths.		Quotients.		Theoretical.	
549.11	549.78	1.1680	1.1666	1.1675	$1 + 3\,b$
548.68	548.75	1.1689	1.1686	1.1691	$1 + 3\,c$
525.98	525.95	1.2194	1.2195	1.2195	$1 + 4\,a$
524.18	524.26	1.2236	1.2234	1.2234	$1 - 4\,b$
523.24	523.23	1.2258	1.2258	1.2255	$1 + 4\,c$
522 97	522.83	1.2264	1.2268	1.2266	$1 + \rho\,d$
522.71	522.60	1.2270	1.2273	1.2272	$1 + \sqrt{2}\,c$

The ratio between the first and the ninth harmonic increment, $\rho = 1.4232$, is my theoretical value for the ratio between heat of constant pressure and heat of constant volume :* the ratio between the second and the tenth harmonic increment, $\sqrt{2}$, is the ratio between dissociative- or wave-velocity, and stable- or circular-velocity. The geometric mean of 1.1645, 1.1680, 1.1689, is $1.1671 = 1 + 3\,b'$; $(1.2194 \times 1.2236 \times 1.2258)^{\frac{1}{3}} = 1.2229 = 1 + 4\,b'$. Huggins's means are not so theoretically exact, but their deviation is far within the limits of probable error, for $(1.1649 \times 1.1666 \times 1.1686)^{\frac{1}{3}} = 1.1668$; $(1.2195 \times 1.2234 \times 1.2258)^{\frac{1}{3}} = 1.2229$; $1 + 3\,b'' = 1.1670$; $1 + 4\,b'' = 1.2227$. Kirchhoff gives the following additional lines :

(2) STRONTIUM, K.

Wave-lengths.	Quotients.	Theoretical.	
650.68	.9857		
554.52	1.1566		
461.69	1.3892	1.3893	$1 + 4\,a'$
461.62	1.3894	1.3898	$1 + 4\,b'$
431.38	1.4868	1.4867	$1 + 5\,a'$
431.18	1.4875	1.4872	$1 + 5\,b'$

PLATINUM, K H.

Wave-lengths.		Quotients.		Theoretical.	
598.22	598.14	1.0000	1.0000	1.0000	
596.86	596.59	1.0023	1.0026	1.0026	$1 + 3\,a$
595.62	595.47	1.0044	1.0045	1.0044	$1 + 5\,a$
548.07	547.95	1.0915	1.0916	1.0910	$1 + 5\,b$
530.70	530.76	1.1272	1.1270	1.1275	$1 + 7\,b$
523.10	523.08	1.1436	1.1435	1.1419	$1 + 7\,c$
506.43	506.32	1.1812	1.1813	1.1825	$1 + 9\,c$
456.19	454.92	1.3113	1.3148	1.3129	$1 + 20\,d$
450.77	449.72	1.3271	1.3300	1.3285	$1 + 21\,d$
445.65	444.45	1.3424	1.3455	1.3442	$1 + 22\,d$

This is not given among the comparisons in Gibbs's Table XI, but it embraces all the lines in which Huggins's measurements (Table IV) and

* Proc. Soc. Phil. Amer., xiv, 651.

Kirchhoff's (Table IX) differ by less than a unit. The groups may be connected by the equations, $21\ a = b$; $10\ b = 9\ c$; $6\ b = 7\ d$.

The foregoing investigations were undertaken in consequence of a suggestion by Professor Henry Draper, that I should test my theory of harmonic undulatory influence by an examination of spectral lines. Professor Asaph Hall led me to the discovery of further corroborative tests, by the query, "Will the inner moon of Mars fall into harmony, or will it make a discord?" *

If we start from a point near the theoretical beginning of nebular condensation for the outer satellite,† and take $2 \times 3 - 1$ harmonic divisors, of the form $\text{div.}_{n+1} = 3\ \text{div.}_n - \text{div.}_1 = \text{div.}_n + 3^{n-1}$, we find the following accordances :

Numerator.	Divisors.	Quotients.	Observed.
13.7	$d_1 = 1$	13.700	13.692 = Nebular radius.
$d_2 = 3\ d_1 - d_1 =$	2	6.850	6.846 = Deimus.‡
$d_3 = 3\ d_2 - d_1 =$	5	2.740	2.730 = Phobus.‡
$d_4 = 3\ d_3 - d_1 =$	14	.979	1.000 = ♂ semi-diam.
$d_5 = 3\ d_4 - d_1 =$	41	.334	.333 = ♂ c. of rad. osc.

In a letter to the editors of the *American Journal of Science and Arts* (Oct., 1877, p. 327), Professor Kirkwood calls attention to the rapid motion of the inner satellite, and asks : "How is this remarkable fact to be reconciled with the cosmogony of Laplace?" He suggests a partial explanation, based upon the motions of Saturn's ring, and concludes with the remark : "Unless some such explanation as this can be given, the short period of the inner satellite will doubtless be regarded as a conclusive argument against the nebular hypothesis."

This is undoubtedly true, if we accept the nebular hypothesis in the form in which it is popularly taught, and in which Laplace is commonly supposed to have held it. But there are probably very few among the students who have given the subject much careful attention, who have supposed that all the planet-building has taken place at the "limit of possible atmosphere," or the point of equal centripetal and centrifugal force. It may well be doubted whether the illustrious French Astronomer ever held such an opinion, and it is certain that Sir William Herschel never did, for he speculated on the "gradual subsidence and condensation" of nebulous matter "by the effect of its own gravity, into more or less regular spherical or spheroidal forms, denser (as they must in that case be) towards the centre." §

As necessary consequences of such subsidence, there would be an acceleration of velocity in all the nebular particles, the acceleration being more rapid in the nucleus, than near the outer surface of the nebula. Many in-

*See Journal of the Franklin Institute, Nov., 1877.

† *Phil. Mag.* Oct., 1877, p. 292.

‡These are the names proposed for the satellites by their discoverer, Prof. Asaph Hall.

§ Herschel's "Outlines of Astronomy," § 871.

dications point to the simultaneous, or nearly simultaneous, initiation of numerous planetary centres, and it is very doubtful if either of the two-planet belts, except, perhaps, that of Neptune and Uranus, will be long regarded as having been "thrown off" by the mere increase of centrifugal velocity.

At the very outset of my own investigations,* I was careful to limit my acceptance of the nebuar hypothesis to the qualified exposition of its originator, as stated by Sir John Herschel: "Neither is there any variety of aspect which nebulæ offer, which stands at all in contradiction to this view. Even though we should feel ourselves compelled to reject the idea of a gaseous or vaporous 'nebulous matter,' it loses little or none of its force. Subsidence, and the central aggregation consequent on subsidence, may go on quite as well among a multitude of discrete bodies, under the influence of mutual attraction, and feeble or partially opposing projectile motions, as among the particles of a gaseous fluid." †

It matters not whether there is such a thing as a luminiferous æther, or whether the hypothesis of such an entity is merely a convenient assumption for the co-ordination of results which are due to the action of forces *such as would exist* in such a medium. The proper study of the forces, and of their mathematical consequences, is the great thing to be sought, and the numerous accordances which I have already found, show how prolific such studies may become. Those accordances, as it seems to me, are already sufficient to establish the Herschelian hypothesis as a true theory, beyond the reach of all possible controversy. That the elastic, or quasi-elastic, forces, which are continually operating throughout the solar system, should extend the harmonic laws to the satellites, as well as to the planets and to the spectral lines, is a necessary consequence of the simplicity and unity of design which underlie the manifold phenomena of the universe.

In the case of our own moon, as we have only two terms, Earth's semi-diameter and Moon's orbital major-axis, the harmonic equation is indeterminate; its direct solution is, therefore, impossible. I have elsewhere, however, called attention to the fact that Earth is central, in the belt which is bounded by the secular perihelion of Mercury and the secular aphelion of Mars, and this fact, together with the nearly synchronous rotation of all the planets in the belt, may be regarded as indications of common forces, such as would be likely to lead to common harmonies. The sixth and seventh divisors of the Mars series represent, respectively, the ratio of Earth's semi-diameter to Moon's major-axis, and the ratio of Earth's axial rotation to its orbital revolution, viz.:

$d_6 = 3\,d_5 - d_1 = 122.$ 120.5331 = Moon's major-axis.

$d_7 = 3\,d_6 - d_1 = 365$ 365.2564 = Earth's year.

The harmonic series, of which Mars and its satellites form a part, seems to have been established before the ring of greatest nebular condensation—the ring of which Earth was the centre—was broken up. In the solar

* *Phil. Mag.*, April, 1876.

† *Loc. cit.*

system, as well as in the group of densest planets, the number 3, which represents the uneven harmonics of an organ-pipe, as well as the oscillatory divisions of a linear pendulum, holds a prominent place. For we find, at the outset, the following approximations to important nebular centres :

$3^8 = 9^4 =$		6561	$6518 =$	Neptune's secular aphelion.
3^7		2187	2222	Saturn's secular aphelion.
3^6	9^3	729	735	Cybele.
3^5		243	229	Earth's secular aphelion.
3^4	9^2	81	83	Mercury.
3^3		27	..	
3^2	9^1	9	..	
3^1		3	..	
3^0	9^0	1	1	Sun's semi-diameter.

This accordance is the more significant, because Saturn's secular aphelion is at the centre of the ring of secondary condensation, which extends from Sun's surface to Uranus's secular aphelion.

"Bode's Law," was based on successive differences of $2^0 \times 3$, $2^1 \times 3$, $2^2 \times 3$, etc. If we substract 1 from each of the theoretical Bode numbers, and divide the remainders by 3, the quotients are 1, 2, 3, 5, 9, 17, etc., each of the quotients, except those for Venus and Neptune, being of the form $d_{n+1} = 2\, d_n - 1$; the dense-belt series being of the form $d_{n+1} = 3\, d_n - 1$.

In the infinite series, $\frac{1}{2} + 3^{-\infty} + 3^{-\infty+1} + \ldots 3^{-1} + 3^0 + 3^1 + 3^2 + \ldots$, successive sums, in the neighborhood of unity, give the following accordances :

Sums.		Harmonic Divisors.	Quotients.	Observed.
$\frac{1}{2}$	=	$\frac{1}{2}$	27.38	$27.00 = 3^3$.
$+ 3^{-\infty}$		.	.	.
+ .		.	.	.
+ .		.	.	.
$+ 3^{-4}$		$\frac{14}{27}$	26.40	26.20 Extreme major-axis.
$+ 3^{-3}$		$\frac{5}{9}$	24.64	24.39 Mean major-axis.
$+ 3^{-2}$		$\frac{2}{3}$	20.53	20.68 Extreme secondary radius.
$+ 3^{-1}$		1	13.69	13.69 Nebular radius.
$+ 3^0$		2	6.85	6.85 Deimus.
$+ 3^1$		5	2.74	2.73 Phobus.
$+ 3^2$		14	.98	1.00 Semi-diameter of Mars.
$+ 3^3$		41	.33	.33 Oscillatory centre.
$+ 3^4$		122		120.56 Moon's major-axis.
$+ 3^5$		365		365.26 Terrestrial acceleration
$+ 3^6$		1094		1096.20 Jupiter's semi-major-axis.

The "Extreme major-axis" is the major-axis of an ellipse, connecting the inner planets of the two outer two-planet belts at the secular aphelia of Uranus and Jupiter ; the "Mean major-axis" is the sum of the mean distances of Uranus and Jupiter ; the "Extreme secondary radius" is Uranus's aphelion radius, or the semi-diameter of the ring of secondary condensation ; the "Nebular radius" not only represents the theoretical incipience of Mars's nebular condensation, but it also corresponds, almost precisely, with the sum of the secular perihelia of Jupiter (4.886) and Saturn (8.734), in units of Earth's semi-major-axis—the secular perihelion being the time of greatest orbital *vis viva;* "Moon's major-axis" is also Earth's "Nebular radius ;" the "Terrestrial acceleration" represents the theoretical increase in the angular velocity of Earth's rotation, since its rupture from the central nucleus, or the ratio of its day to its year ; "Jupiter's semi-major-axis" is measured in units of Sun's mean perihelion distance from the centre of gravity of Sun and Jupiter.

The sum of the infinite series, to and including 3^{-3}, is $\frac{5}{9}$, which represents the ratio of *vis viva* between undulatory velocity and the velocity of the particles of a medium constituted according to the Kinetic theory * Alexander has shown the importance of that ratio in planeto-taxis,† and I have shown that it represents "centres of explosive oscillation," or the centre of secondary oscillation between the primary centre of oscillation and the centre of gravity, in a homogeneous line of particles ($\frac{2}{3} - \frac{2}{3}$ of $\frac{1}{6}$ $= \frac{5}{9}$). Adding the next term of the series, we get $\frac{2}{3}$, which represents the centre of linear oscillation. Neptune's major-axis (60.06) is, within $\frac{1}{10}$ of 1 per cent., ($3^4 - 3^3 + 3^2 - 3^1 = 60$) times Earth's mean radius vector.

These harmonies embrace orbital radii of the largest five planets of the solar system, of the inner planets, and of the asteroidal belt, together with nebular-, satellite-, and planetary-radii, for the outer and the middle planets in the theoretically primitive central belt, or the belt of greatest condensation. Can any interpretation be rightly put upon such a chain of harmonies, which does not recognize the fundamental laws of harmonic oscillation and harmonic design?

Neither of Mars's moons is of sufficient magnitude to cause any great perturbations. To this fact, perhaps, as much as to the proximity of the density-centre, we may attribute the regularity of the Mavortian system. In the solar system, as we have seen,‡ the preponderating mass of Jupiter sets up a new order of differences in the harmonic denominators ; and we may find probable indications of similar influence in some of the satellite systems, and in the elementary spectra.

In the satellite system of Uranus, if we take the semi-major-axis of the outer satellite as the common numerator (22.75), we find the following harmony :

Satellites.	Distances.	Denominators.	Theoretical.
Oberon,	22.75	1.000	1.000
Titania,	17.01	1.337	$1.343 = 1 + 2a$
Umbriel,	10.37	2.194	$2.199 = 1 + 7a$
Ariel,	7.44	3.058	$3.055 = 1 + 12a$
Semi-diameter,	1.00	22.750	$22.750 = 1 + 127a$

* Maxwell and Preston, *Phil. Mag.*, June, 1877.

† Smithsonian Contributions, 280.

‡ *Ante*, xii, 403*sqq.* ; xiii, 237-9 ; etc.

In the Saturnian system there is a slight uncertainty in the satellite elements, except in the case of Titan, whose orbit was well determined by Bessel. It will be seen that Titan's great mass introduces a secondary harmony. The following harmonic denominators are based upon relative mean distances which would represent the orbital times, as furnished by Professor Hall :

Satellites.	Times.	Denominators.	Theoretical.	
Japetus,	79.3292^d	1.000	1.000	
Hyperion,	21.3113	2,402	2.397 =	$1 + a$
Titan,	15.9454	2.914	2.920	$1 + b$
Rhea,	4.5175	6.756	6.760	$1 + 3b$
Dione,	2.7369	9.436	9.384	$1 + 6a$
Tethys,	1.8878	12.087	12.179	$1 + 8a$
Enceladus,	1.3702	14.966	14.974	$1 + 10a$
Mimas.	.9425	19.206	19.166	$1 + 13a$
Semi-diameter,		64.359	64.360	$1 + 33b$

It is well to notice that b (1.920) is very nearly the square of a (1.397).

In the column of times, Japetus, divided by Titan, is nearly 5 ; Hyperion, by Rhea, 5 ; Dione, by Enceladus, 2 ; Tethys, by Mimas, 2 ; Titan, by Rhea, $\frac{7}{2}$: Rhea, by Dione, $\frac{5}{3}$; Hyperion, by Titan, $\frac{4}{3}$; Hyperion, by Dione, 8 ; Hyperion, by Mimas, $\frac{45}{2}$: Titan, by Mimas, 17.

The satellite system of Jupiter, our Sun's "companion star," exhibits harmonies of distance, time and mass. The mean distance of the outer satellite, Callisto, is 3^3 semi-diameters of its primary (26.9984). Using this as a common numerator, we find that the other satellites are phyllotactically, as well as harmonically, arranged :

Satellites.	Distances.	Denominators.	Theoretical.	
Callisto,	26.9984	1.000		
Ganymede,	15.3502	1.759	1.731 =	$5a$.
Europa,	9.6235	2.807	2.769	$8a$.
Io,	6.0485	4.464	4.500	$13a$.
Semi-diameter,	1.0000	26.998	26.998	$78a$.

The harmonies of time and mass are as follows :

Satellites.	Times.	Theoretical.		Mass.	Theoretical.	
Callisto,	16.689^d	16.684 =	$28t$	4266	4403 =	$\frac{1}{2}m$.
Ganymede,	7.155	7.150	$12t$	8850	8806	$1m$.
Europa,	3.551	3.575	$6t$	2324	2202	$\frac{1}{4}m$.
Io.	1.769	1.788	$3t$	1733	1761	$\frac{1}{5}m$.

The interesting and valuable communications of Professor Alexander, to the last semi-annual meeting of the National Academy, exhibit various harmonies in the several satellite systems, some of which are closer than my own, others are the same, and others are not so close. He recognizes the important influences of linear centres ($\frac{1}{2}$), centres of linear oscillation ($\frac{2}{3}$, $\frac{1}{3}$), centres of atmospheric dissociation $\left(\frac{n}{n+1}\right)$, mean or extreme apsidal distances, mean eccentricities, and a resisting medium, to all of which I called attention five or six years ago.* He thus obtains a planetary series of great symmetry and beauty, but it is neither so close in its general approximations, so broad in its indications, nor so simple in its law, as my series of harmonic nodes, determined by the overshadowing influence of Jupiter.† His figures, however, in connection with my own, show that the law of simple harmonic interferences is universally operative, between adjacent planets and satellites, as well as in the systematic subordination of whole groups to more widely controlling masses.

I quite agree with Professor Alexander, in thinking that the relations of the mean distances, detailed in his "Harmonies," ‡ belong to a very ancient and probably formative state of the system ; while those of the extreme distances, as also Stockwell's curious relations between the perihelia and nodes of the outer planets, § have been brought about by subsequent perturbations. According to the nebular hypothesis, we might naturally look, when rotation was first established, for arrangements determined by centres of spherical gravity, inertia and oscillation. But as soon as nucleal points appeared, corresponding linear centres began to be operative, and their influence must have become more and more prevalent as condensation went on, leading to the many consequences which I have already pointed out, as well as to many others, the discovery of which will doubtless reward the labors of future investigators. Evidences of perturbative action originating since the establishment of the terrestrial nucleus, seem to be given by the following equations :

$$\frac{n}{n_1} = \frac{\partial_1}{\partial} \tag{7}$$

$$\left(\frac{\partial_2}{\partial_1}\right)^3 (f_1 r_1)^{\frac{1}{2}} = \lambda \tag{8}$$

In these equations n_1 = the special coefficient of Jupiter's dissociative velocity ($n_1 \pi^{-1} \sqrt{f_1 r_1}$) : ∂_1 = Jupiter's secular perihelion distance from the Sun ; ∂_2 = Uranus's mean distance from Sun ; $(f_1 r_1)^{\frac{1}{2}}$ = limit of satellite-velocity at Jupiter. In view of the many pointings which we thus find towards the limiting velocity of light, it seems probable that the solar-dissociative velocity is still continually efficient, through the combined influences of virtual fall and elasticity, in maintaining the gaseous structure of the Sun. Alexander's relations between Saturn's moons and belts indicate a similar gaseous structure in the belted planet ; but even in the Saturnian system my harmonic series gives closer approximations to actual lunar distances, except in the cases of Titan and Tethys, than Alexander's series, which represents centres of atmospheric dissociation, thus doubly confirming the hypothesis that centres of spheroidal activity are first operative, and that afterwards, linear centres modify and extend the primitive harmonies. Titan is Saturn's giant moon. The ratio of distance to planetary radius, for Tethys, is the same as the ratio between the limiting satellite-velocities of Jupiter and Earth.

*Ante, vols. xii, 403–7, 412, 520; xiii, 146, 196 (11); xiv, 655, etc.
†Ante, xiii, 196 (11); 237–9.
‡Smithsonian Contributions, 280.
§Smithsonian Contributions, 232, p. xiv.